Physics of fluids, tentamen 8 April, 2014

Question 1

a) A stream line is a curve that is instantaneously tangent to the fluid velocity throughout the domain (see Fig. 3.5 of Kundu).

A path line is the trajectory of a fluid particle of fixed identity (see Fig. 3.4 Kundu)

4 A streak line is the curve obtained by connecting all fluid particles that will pass or have passed through a fixed point in space

b) $\frac{dx}{u} = \frac{dy}{v} \Longrightarrow \frac{dy}{dx} = \frac{v}{u} = \frac{v_0 \sin(\omega t - kx)}{u_0}$ (2)

6 $\Rightarrow \int u_0 \, dy = \int v_0 \sin(\omega t - kx) dx = -\frac{v_0}{k} \int \sin(\omega t - kx) \, d(\omega t - kx)$

$\Rightarrow \quad u_0 y = \frac{v_0}{k} \cos(\omega t - kx) + C$ (2)

$x = y = 0$ for $t = 0$ $\Rightarrow C = -\frac{v_0}{k} \Rightarrow y = \frac{v_0}{u_0 k}\left(\cos(kx) - 1\right)$ (2)

c) $u = \frac{dx}{dt}$, $v = \frac{dy}{dt}$

$\hookrightarrow x = u_0 t + C$ since $x = 0$ at $t = 0 \Rightarrow C = 0$

$$dy = V_0 \sin(\omega t - kx)\,dt \underset{x = u_0 t}{=} V_0 \sin((\omega - k u_0)t)\,dt$$

$$\Rightarrow \quad y = \frac{-V_0}{\omega - k u_0} \cos((\omega - k u_0)t) + C$$

6 since $y=0$ at $t=0$ $\Rightarrow$ $C = \dfrac{V_0}{\omega - k u_0}$. Substituting $t = \dfrac{x}{u_0}$

yields: $\quad y = \dfrac{-V_0}{\omega - k u_0}\left(\cos\left(\left(\dfrac{\omega - k u_0}{u_0}\right)x\right) - 1\right)$, $\omega - k u_0 \neq 0$

d) The flow lines coincide for steady flows,
2 i.e., for $\omega = 0$.

<u>Question 2</u> a) $$\frac{d}{dt}\int_{V(t)} \rho \zeta \, dV + \int_{A(t)} \underline{\Theta} \cdot \underline{n} \, dA \underset{\text{Reynolds \& Gauss}}{=}$$

$$= \int_{V(t)} \frac{\partial}{\partial t}(\rho \zeta)\, dV + \int_{A(t)} \rho \zeta (\underline{u} \cdot \underline{n})\, dA + \int_{V(t)} \underline{\nabla} \cdot \underline{\Theta}\, dV \underset{\text{Gauss}}{=}$$

$$= \int_{V(t)} \underbrace{\left\{ \frac{\partial}{\partial t}(\rho \zeta) + \underline{\nabla} \cdot (\rho \zeta \underline{u}) + \underline{\nabla} \cdot (-\rho \underline{\nabla} \zeta) \right\}}_{=0} dV = 0$$

$\Rightarrow$

$$\rho \frac{\partial}{\partial t}\zeta + \zeta\left(\frac{\partial \rho}{\partial t} + \rho \underline{u}\cdot\nabla\zeta + \zeta\nabla\cdot(\rho\underline{u}) - \nabla\cdot(\rho\nabla\zeta)\right.$$

$$= \zeta\underbrace{\left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u})\right)}_{=0} + \rho\frac{\partial \zeta}{\partial t} + \rho\underline{u}\cdot\nabla\zeta - \nabla\cdot(\rho\nabla\zeta) = 0$$

$$\Rightarrow \frac{\partial \zeta}{\partial t} + \underline{u}\cdot\nabla\zeta = \frac{1}{\rho}\nabla\cdot(\rho\nabla\zeta)$$

Question 3

(4) a) Incompressibility: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$u = \frac{\partial \phi}{\partial x} = 2Ax + \frac{m}{2\pi}\frac{x}{x^2+y^2} \qquad v = \frac{\partial \phi}{\partial y} = -2Ay + \frac{m}{2\pi}\frac{y}{x^2+y^2}$$

$$\left.\begin{aligned}\frac{\partial^2\phi}{\partial x^2} &= 2A + \frac{m}{2\pi}\left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right)\\ \frac{\partial^2\phi}{\partial y^2} &= -2A + \frac{m}{2\pi}\left(\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right)\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2A - 2A + \frac{m}{2\pi}\left(\frac{2}{x^2+y^2} - \left(\frac{2x^2+2y^2}{(x^2+y^2)^2}\right)\right)$$

$$= \frac{m}{2\pi}\left(\frac{2}{x^2+y^2} - \frac{2(\cancel{x^2+y^2})}{(x^2+y^2)^{\cancel{2}}}\right) = 0$$

(2) Irrotational: $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x} = 0$

b) $\phi = A(x^2-y^2) + \frac{m}{2\pi} \ln\sqrt{x^2+y^2}$

$$= Ar^2(\cos^2\theta - \sin^2\theta) + \frac{m}{2\pi} \ln r$$

$x = r\cos\theta$
$y = r\sin\theta$

$$u_r = \frac{\partial\phi}{\partial r} = 2Ar(\cos^2\theta - \sin^2\theta) + \frac{m}{2\pi r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$\Rightarrow \frac{\partial\psi}{\partial\theta} = 2Ar^2(\cos^2\theta - \sin^2\theta) + \frac{m}{2\pi}$$

$$\Rightarrow \psi = \int 2Ar^2\cos(2\theta)\,d\theta + \frac{m\theta}{2\pi} + f(r)$$

$$\psi = \cancel{2}Ar^2 \cdot \frac{1}{\cancel{2}}\sin(2\theta) + \frac{m\theta}{2\pi} + f(r)$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(Ar^2\cos 2\theta + \frac{m}{2\pi}\ln r\right)$$

$$= -\frac{1}{\cancel{r}}Ar^{\cancel{2}}2\sin 2\theta = -2Ar\sin 2\theta = -\frac{\partial\psi}{\partial r}$$

$$\Rightarrow \frac{\partial\psi}{\partial r} = 2Ar\sin 2\theta \qquad \Rightarrow \psi = \cancel{2}A\sin 2\theta \cdot \frac{1}{\cancel{2}}r^2 + g(\theta)$$

$$\Rightarrow \psi = Ar^2\sin 2\theta + \frac{m\theta}{2\pi} \qquad \left(f(r) = 0,\ g(\theta) = \frac{m\theta}{2\pi}\right)$$

c) $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}\left(2Ar^2\cos 2\theta + \frac{m}{2\pi}\right)$

$$u_r = 2Ar\cos 2\theta + \frac{m}{2\pi r}$$

$$u_\theta = -\frac{\partial \psi}{\partial r} = -2Ar\sin(2\theta) \qquad \Rightarrow \sin(2\theta) = 0$$

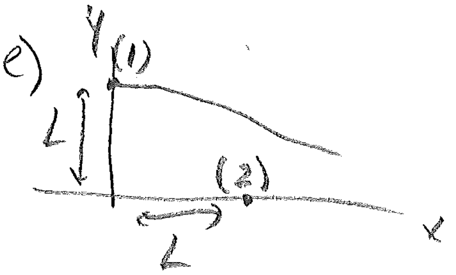
$$\Rightarrow \theta = \frac{\pi}{2}$$

$$\Rightarrow u_r = 2Ar\underbrace{\cos(\pi)}_{-1} + \frac{m}{2\pi r} = 0 \quad \Rightarrow 2Ar = \frac{m}{2\pi r}$$

$$\Rightarrow m = 4\pi A r^2 \quad \Rightarrow r = \sqrt{\frac{m}{4\pi A}} = L$$

stagnation point: $(r,\theta) = \left(\sqrt{\frac{m}{4\pi A}}, \frac{\pi}{2}\right)$

d) $\psi\left(\sqrt{\frac{m}{4\pi A}}, \frac{\pi}{2}\right) = A\frac{m}{4\pi A}\cdot\underbrace{\sin\pi}_{=0} + \frac{m}{2\cancel{\pi}}\frac{\cancel{\pi}}{2} = \frac{m}{4}$

e) 

$$P_1 + \frac{1}{2}\rho \overbrace{v_1^2}^{=0} = P_2 + \frac{1}{2}\rho v_2^2$$

$$P_1 - P_2 = \frac{1}{2}\rho v_2^2 = \frac{1}{2}\rho\left(u_r^2 + u_\theta^2\right)$$

$$u_r\Big|_{\substack{\theta=0\\ r=L}} = 2AL + \frac{m}{2\pi L} \qquad u_\theta\Big|_{\substack{\theta=0\\ r=L}} = 0$$

$$\Rightarrow u_r = 2AL + \frac{m}{2\pi L} \underset{\substack{\uparrow\\ m = 4\pi A L^2}}{=} 2AL + \frac{4\pi A L^2}{2\pi L} = 4AL$$

$$\Rightarrow P_1 - P_2 = \frac{1}{2}\rho\cdot 16A^2L^2 = 8\rho A^2 L^2$$

Question 4

a) $\cancel{\frac{\partial \rho}{\partial t}} + \frac{\partial}{\partial x_i}(\rho u_i) = \rho \frac{\partial u_i}{\partial x_i} = \rho\left(\cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y}\right) = 0$

steady — fully developed

$\Rightarrow \quad \frac{\partial v}{\partial y} = 0 \qquad v = \text{constant} = 0$

$v(x=0) = 0$

$\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad u = u(y)$

b) $\frac{Du_j}{Dt} = -\frac{\partial p}{\partial x_j} + \rho g_j + \mu \frac{\partial^2 u_j}{\partial x_i^2} = \rho\left(\cancel{\frac{\partial u_j}{\partial t}} + u_i \frac{\partial u_j}{\partial x_i}\right)$

steady

$i=1$: $\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

$0 = -\underbrace{\cancel{\frac{\partial p}{\partial x}}}_{=0} + \rho g_x + \mu \frac{d^2 u}{dy^2}$

$i=2$: $\quad 0 = -\frac{\partial p}{\partial y} + \rho g_y$

$i=3$: $\quad 0 = 0$

$\Rightarrow \quad i=1: \quad 0 = \rho g \sin\theta + \mu \frac{d^2 u}{dy^2} \quad \Rightarrow \quad \frac{d^2 u}{dy^2} = \underbrace{\frac{-\rho g \sin\theta}{\mu}}_{\alpha}$

$\frac{du}{dy} = \alpha y + C_1 \quad \Rightarrow \quad u(y) = \frac{1}{2}\alpha y^2 + C_1 y + C_2$

B.C.1: $\tau = \mu \left.\frac{du}{dy}\right|_{y=h} = 0 \quad \Rightarrow \quad \alpha h + C_1 = 0 \quad \Rightarrow \quad C_1 = -\alpha h$

B.C.2 : $u(y=0) = -U \Rightarrow C_2 = -U$

$$\Rightarrow \quad u(y) = \frac{1}{2}\alpha y^2 - \alpha h y - U \quad = -\frac{\rho g \sin\theta}{\mu}\left(\frac{y^2}{2} - hy\right) - U$$

$$Q = \int_0^h u(y)\,dy = \left[\frac{1}{6}\alpha y^3 - \frac{\alpha h}{2} y^2 - Uy\right]_0^h$$

$$= \frac{1}{6}\alpha h^3 - \frac{\alpha h^3}{2} - Uh = -\frac{1}{3}\alpha h^3 - Uh$$

$$= \frac{1}{3}\frac{\rho g \sin\theta}{\mu} \cdot h^3 - Uh$$

$$V = \frac{Q}{h} = \frac{\rho g \sin\theta\, h^2}{3\mu} - U$$

c) No drainage : $u(y=h) < 0$

$$-\frac{\rho g \sin\theta}{\mu}\left(\frac{h^2}{2} - h^2\right) - U < 0$$

$$\frac{\rho g \sin\theta}{\mu} \cdot \frac{h^2}{2} < U$$

$\rho \uparrow \rightarrow$ more drainege, so $U \uparrow$

$\mu \uparrow \rightarrow$ less drainage, so $U \downarrow$